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ON THE UNILATERAL CONTACT PROBLEM OF STRUCTURES WITH A NON QUADRATIC STRAIN ENERGY DENSITY

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Abstract—The analysis of structures with "unilateral contact" boundary conditions is considered. The stress–strain relations are nonlinear and they are derived from a non quadratic strain energy density by "subdifferentiation". It is proved that for the inequality constrained boundary value problem the "principles" of virtual and of complementary virtual work hold in an inequality form constituting a variational inequality. The theorems of minimum potential and complementary energy are proved to be valid to account for this type of boundary conditions. These theorems are used to formulate the analysis as a nonlinear programming problem. A numerical example of a structure having the "unilateral contact" boundary condition illustrates the theory.

INTRODUCTION

In the analysis of structures, sometimes one encounters situations, where the boundary presents a unilateral contact with a support. By the term "unilateral contact" it is meant that it is not known *a priori* which part of the boundary works itself loose from the support and which part remains in contact with it. Such boundary conditions resulting from the unilateral contact of a linearly elastic body with a rigid support have been already considered by Signorini and Fichera[2], who developed the abstract mathematical theory of the resprective boundary value problem (B.V.P.).

A more general form of unilateral contact boundary conditions is considered in this paper. They arise, when a body is in unilateral contact with an inelastic support. Moreover the strain energy density of the considered body is assumed to be a convex (generally non quadratic) function of the strain tensor components. This type of strain energy density, already considered by the French school[5, 7] gives rise to a subdifferential constitutive relation and permits us to formulate compactly a large class of nonlinear constitutive laws.

Two variational inequalities generalizing the "principle of virtual work" and of "complementary virtual work" will be obtained and the corresponding minimum potential and complementary energy theorems will be proved. It is shown, that using these theorems, the analysis reduces to a nonlinear programming problem, and accordingly the nonlinear programming algorithms are used for the numerical calculation.

The considered type of boundary condition appears in all the foundation structures. For example it is known that often large footings, mats, retaining walls, pile foundations etc. work themselves partly loose from the soil. Accordingly their stress and displacement fields can be satisfactorily obtained by means of the developed theory.

2. THE BOUNDARY VALUE PROBLEM

Consider in the orthogonal cartesian system $Ox_1x_2x_3$ body Ω with boundary Γ. The total boundary is made up of three nonoverlapping parts denoted Γ_U, Γ_F, Γ_S. On Γ_U (resp. Γ_F) the displacements (resp. the surface forces) have the given values U_i (resp. F_i) and on part Γ_S the unilateral contact boundary conditions hold.

If n_i is the normal outward vector on the boundary and σ_{ij} is the stress tensor, then $S_i = \sigma_{ij}n_j$ $(i, j = 1, 2, 3)$—summation convention—is the resultant of the boundary stresses. To define the boundary condition on Γ_S, this resultant is decomposed into a factor S_{Ni} normal to the boundary and a vector S_{Ti} tangential to the boundary. The algebraic values of these vectors are denoted by S_N and S_T. Further the algebraic value of S_{Ni} is considered to be positive if S_{Ni} is directed outwards of the boundary. Similarly the vector u_i of the boundary Γ_S is decomposed into a normal vector u_{Ni} and a tangential vector u_{Ti} with algebraic values u_N and u_T respectively. The

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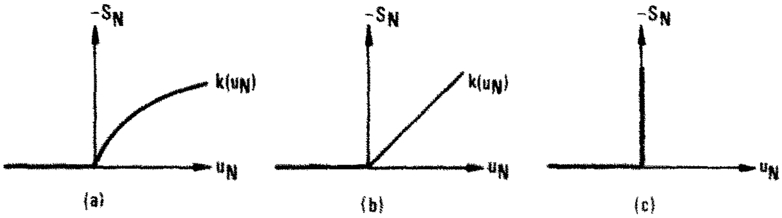


Fig. 1. Forms of function $k(u_N)$.

unilateral contact boundary condition with respect to an inelastic support are given in Fig. 1(a) and have the following form[9]:

$$\text{If } u_N < 0, \quad \text{then } S_N = 0, \tag{1}$$

$$\text{if } u_N \geq 0, \quad \text{then } S_N + k(u_N) = 0. \tag{2}$$

Condition (1) expresses the fact, that when the body works itself loose from the support, then S_N is equal to zero. Condition (2) holds in the regions of Γ_S, where the body is in contact with the support and in this case the law relating u_N with S_N is expressed by means of the equation

$$S_N + k(u_N) = 0, \tag{3}$$

where $k(u_N)$ is a nondecreasing function, which is zero for negative values of u_N.

Figure 1(b) corresponds to the conditions:

$$\text{If } u_N < 0, \quad \text{then } S_N = 0, \tag{4}$$

$$\text{if } u_N \geq 0, \quad \text{then } S_N + ku_N = 0, \tag{5}$$

expressing the unilateral contact with an elastic support, and Fig. 1(c) to the conditions:

$$\text{If } u_N < 0, \quad \text{then } S_N = 0, \tag{6}$$

$$\text{if } u_N = 0, \quad \text{then } S_N \leq 0, \tag{7}$$

expressing the unilateral contact with a rigid support. To formulate a complete boundary value problem, conditions (1) and (2) should be combined with the boundary condition

$$S_{T_i} = C_{T_i} \quad i = 1, 2, 3 \tag{8}$$

on Γ_S, where C_{T_i} is a prescribed function on Γ_S.

Further, under the assumption of small displacements, the equations of equilibrium and the strain displacement relations are respectively

$$\sigma_{ij,j} + f_i = 0 \tag{9}$$

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{10}$$

where ϵ_{ij} is the strain tensor, f_i is the vector of the body forces and the subscript after the comma denotes differentiation. The constitutive relations are written in the form

$$\sigma_{ij} \in \vartheta w(\epsilon), \tag{11}$$

where $\epsilon = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}\}$, w denotes the strain energy density of the body and ϑ is the symbol of subdifferentiation[1–10]. w is assumed to be a convex, lower semicontinuous† strictly positive definite function defined on the space E of the symmetric strain tensors.

†A real valued function f defined over an arbitrary set C of R^n is said to be lower semicontinuous at a point x of C, if

$$f(x) \leq \varliminf_{i\to\infty} f(x_i)$$

for every sequence $\{x_i\}$ in C, such that $x_i \to x$ and the limit of $\{f(x_i)\}$ exists in $[-\infty + \infty]$.

Function w has values in the interval $(-\infty + \infty]$[5]. Relation (12) expresses that σ_{ij} is a subgradient[1] of w at ϵ, i.e. $w(\epsilon)$ is finite ($w(\epsilon) < \infty$) and the variational inequality

$$w(e) - w(\epsilon) \geq \sigma_{ij}(e_{ij} - \epsilon_{ij}) \tag{12}$$

holds for every e in E. The set of all subgradients of w at ϵ is the subdifferential set $\vartheta w(\epsilon)$ of w at ϵ, i.e.

$$\vartheta w(\epsilon) = \{\sigma_{ij} | w(e) - w(\epsilon) \geq \sigma_{ij}(e_{ij} - \epsilon_{ij}) \quad \forall e \in E\}. \tag{13}$$

If $w(\epsilon)$ is finite and differentiable at ϵ, the set $\vartheta w(\epsilon)$ contains only one element, which is the gradient of $w(\epsilon)$ (case of classical elasticity). At the points of nondifferentiability of w, σ_{ij} belongs to the contingent set of w at ϵ [1], i.e. to the set of all possible limits

$$\lim_{e_{ij} \to \epsilon_{ij}} [w(e) - w(\epsilon)]/(e_{ij} - \epsilon_{ij}).$$

This limit gives at the differentiable points the usual derivative. A very useful notion for the derivations of the duality between potential and complementary energy is the notion of the conjugate function w^c [10]. It is defined on the space Σ of the symmetric stress tensors σ_{ij} by means of the relation

$$w^c(\sigma) + w(\epsilon) = \sigma_{ij}\epsilon_{ij}, \tag{14}$$

where $\sigma = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}\}$ and σ and ϵ satisfy (11). Function w^c is the complementary energy density of the structure[6] and has the same properties as w. It is obvious, that instead of (11) relation

$$\epsilon_{ij} \in \vartheta w^c(\sigma) \tag{15}$$

can be used.

We can define the strain energy (resp. the complementary energy) of the structure by means of

$$W(\epsilon) = \begin{cases} \iint_\Omega w(\epsilon)\,\mathrm{d}\Omega & \text{if } w(\epsilon) \text{ is integrable} \\ \infty & \text{otherwise.} \end{cases} \tag{16}$$

(resp.)

$$W^c(\sigma) = \begin{cases} \iint_\Omega w^c(\sigma)\,\mathrm{d}\Omega & \text{if } w^c(\sigma) \text{ is integrable} \\ \infty & \text{otherwise.} \end{cases} \tag{17}$$

It is proved in [5] that (11) describes the behaviour of a large class of materials. We mention here the case of holonomic elastoplasticity, the case of locking materials, all the polygonal stress–strain laws etc.

3. THE VARIATIONAL INEQUALITY OF VIRTUAL WORK AND THE MINIMUM OF POTENTIAL ENERGY

A field X^* of displacements and strains u_i^*, ϵ_{ij}^* is defined to be kinematically admissible if it satisfies the strain-displacement relations (10), the kinematical boundary conditions on Γ_U and the kinematical conditions imposed by the boundary conditions on Γ_S. For example in the case of Fig. 1(c) the boundary conditions on Γ_S introduce the inequality $u_N^* \leq 0$. The actual displacements and strains at the position of equilibrium are denoted by u_i, ϵ_{ij} and thus the differences $u_i^* - u_i$, $\epsilon_{ij}^* - \epsilon_{ij}$ represent the kinematically admissible variations. Moreover σ_{ij}^* denotes the stress fields obtained from ϵ_{ij}^* by means of relation (11). From eqns (9) and (10), the conditions $u_i = U_i$ on Γ_U, and the well known divergence theorem, the identity

$$\int_\Omega \sigma_{ij}(\epsilon_{ij}^* - \epsilon_{ij})\,\mathrm{d}\Omega = \int_\Omega f_i(u_i^* - u_i)\,\mathrm{d}\Omega + \int_{\Gamma_S} [S_N(u_N^* - u_N) + S_{T_i}(u_{T_i}^* - u_{T_i})]\,\mathrm{d}\Gamma$$
$$+ \int_{\Gamma_F} S_i(u_i^* - u_i)\,\mathrm{d}\Gamma \quad \forall\, u_i^*, \epsilon_{ij}^* \in X^* \tag{18}$$

is obtained. Further the positive and negative parts of u_N^*, i.e.

$$u_N^* = \underset{+}{u_N^*} - \underset{-}{u_N^*}, \quad \underset{+}{u_N^*} = \frac{u_N^* + |u_N^*|}{2} \geq 0, \quad \underset{-}{u_N^*} = \frac{-u_N^* + |u_N^*|}{2} \geq 0 \tag{19}$$

are introduced. It can be proved trivialy that relations (1), (2) yield the inequality

$$S_N(u_N^* - u_N) + k(\underset{+}{u_N})(\underset{+}{u_N^*} - \underset{+}{u_N}) \geq 0 \quad \forall\, u_i^* \in X^*. \tag{20}$$

Indeed for $u_N < 0$, $S_N = 0$ and thus inequality (20) is satisfied; if $u_N \geq 0$, then $u_N = u_N$ and thus (20) becomes $\underset{+}{u_N^*} \geq u_N^*$, which is valid. Thus eqn (18) yields by means of the boundary conditions on Γ_F the variational inequality

$$\int_\Omega \sigma_{ij}(\epsilon_{ij}^* - \epsilon_{ij})\,\mathrm{d}\Omega + \int_{\Gamma_S} [k(\underset{+}{u_N})(\underset{+}{u_N^*} - \underset{+}{u_N}) - C_T(u_{T_i}^* - u_{T_i})]\,\mathrm{d}\Gamma$$
$$- \int_{\Gamma_F} F_i(u_i^* - u_i)\,\mathrm{d}\Gamma - \int_\Omega f_i(u_i^* - u_i)\,\mathrm{d}\Omega \geq 0 \quad \forall\, u_i^* \in X^*. \tag{21}$$

Conversely it can be proved using the methods indicated in [1], that inequality (20) yields the equation of equilibrium and the boundary conditions on Γ_F and on Γ_S. Indeed choosing the special variations $u_i^* - u_i = 0$ on Γ and $\pm(u_i^* - u_i)$ in the interior of Ω, (21) yields by means of the divergence theorem the equation

$$\int_\Omega (\sigma_{ij,j} + f_i)(u_i^* - u_i)\,\mathrm{d}\Omega = 0, \tag{22}$$

which by means of the well known lemma of the calculus of variations, gives eqn (9). Strictly speaking eqn (22) is a weak formulation of eqn (9). From eqn (9), identity (18) can be obtained; relations (18) and (21) yield the variational inequality

$$\int_{\Gamma_S} [S_N(u_N^* - u_N) + k(\underset{+}{u_N})(\underset{+}{u_N^*} - \underset{+}{u_N}) + (S_{T_i} - C_{T_i})(u_{T_i}^* - u_{T_i})]\,\mathrm{d}\Gamma$$
$$+ \int_{\Gamma_F} (S_i - F_i)(u_i^* - u_i)\,\mathrm{d}\Gamma \geq 0 \quad \forall u_i^* \in X^*. \tag{23}$$

From relation (23) with the help of the special variations $u_i^* - u_i = 0$ on Γ_S and $\pm(u_i^* - u_i)$ on Γ_F the boundary condition

$$S_i = F_i \text{ on } \Gamma_F \tag{24}$$

results. Similarly by taking $u_N^* - u_N = 0$ and $\pm(u_{T_i}^* - u_{T_i})$ on Γ_S, condition (8) is obtained. Now the remaining variational inequality

$$\int_{\Gamma_S} [S_N(u_N^* - u_N) + k(\underset{+}{u_N})(\underset{+}{u_N^*} - \underset{+}{u_N})]\,\mathrm{d}\Gamma \geq 0 \quad \forall\, u_i^* \in X^* \tag{25}$$

will be examined. It will be proved, that inequality (25) yields the boundary conditions (1) and (2) on Γ_S. Indeed for $u_N < 0$, $k(\underset{+}{u_N})$ equals to zero and accordingly, inequality (25) gives for $\pm(u_N^* - u_N)$ the condition $S_N = 0$. For $u_N \geq 0$ substitute in (25)

$$u_N^* = \rho \bar{u}_N, \quad \bar{u}_N \geq 0, \quad \rho \gtreqless 0 \tag{26}$$

and by dividing through ρ and for $\rho \to \infty$ the inequality

$$\int_{\Gamma_S} [S_N \bar{u}_N + k(\underset{+}{u_N})\underset{+}{\bar{u}_N}]\, \mathrm{d}\Gamma \geq 0, \quad \forall \bar{u}_N \geq 0 \tag{27}$$

results. From (27) follows the pointwise inequality

$$S_N + k(\underset{+}{u_N}) \geq 0. \tag{28}$$

Inequality (25) yields for $u_N^* = 0$.

$$\int_{\Gamma} [S_N u_N + k(\underset{+}{u_N})\underset{+}{u_N}]\, \mathrm{d}\Gamma \leq 0, \tag{29}$$

which with (27) gives the remaining boundary condition (2) on Γ_S. Accordingly the variational inequality (21) characterizes the position of equilibrium and expresses the principle of virtual work for the unilateral contact boundary value problem. It contains the nondifferentiable term, expressing the virtual work of the reactions of the unilateral support. This term includes the positive functions $\underset{+}{u_N}$, causing the variations to be unilateral and accordingly the principle of virtual work has an inequality form [4]. To prove the theorem of the minimum of potential energy, the function

$$K(\xi) = \int_0^{\xi} k(\xi)\, \mathrm{d}\xi \tag{30}$$

is introduced. $K(\xi)$ is convex, because of the monotonicity of $k(\xi)$, and thus [1] the inequality

$$K(\underset{+}{u_N^*}) - K(\underset{+}{u_N}) \geq (\underset{+}{u_N^*} - \underset{+}{u_N})k(\underset{+}{u_N}) \quad \forall\, u_i^* \in X^* \tag{31}$$

is valid. Moreover the inequality

$$W(\epsilon^*) - W(\epsilon) \geq \int_{\Omega} (\epsilon_{ij}^* - \epsilon_{ij})\sigma_{ij}\, \mathrm{d}\Omega \tag{32}$$

arising from (11), (12) and (16) is used further.

By means of relations (31) and (32), inequality (21) becomes

$$\begin{aligned} W(\epsilon^*) + \int_{\Gamma_S} K(\underset{+}{u_N^*})\, \mathrm{d}\Gamma - \int_{\Gamma_S} C_{T_i} u_{T_i}^*\, \mathrm{d}\Gamma - \int_{\Gamma_F} F_i u_i^*\, \mathrm{d}\Gamma - \int_{\Omega} f_i u_i^*\, \mathrm{d}\Gamma \\ \geq W(\epsilon) + \int_{\Gamma_S} K(\underset{+}{u_N})\, \mathrm{d}\Gamma - \int_{\Gamma_S} C_{T_i} u_{T_i}\, \mathrm{d}\Gamma - \int_{\Gamma_F} F_i u_i\, \mathrm{d}\Gamma \\ - \int_{\Omega} f_i u_i\, \mathrm{d}\Gamma, \quad \forall u_i^*, \epsilon_{ij}^* \in X^*. \end{aligned} \tag{33}$$

Accordingly any solution of the unilateral contact boundary value problem minimizes at the position of equilibrium the potential energy

$$\Pi = W(\epsilon) + \int_{\Gamma_S} K(\underset{+}{u_N})\,d\Gamma - \int_{\Gamma_S} C_{T_i} u_{T_i}\,d\Gamma - \int_{\Gamma_F} F_i u_i\,d\Gamma - \int_{\Omega} f_i u_i\,d\Omega, \tag{34}$$

over the set of admissible displacement and strain fields. Conversely the nonnegativity of the first variation of (34) yields the variational inequality (21). For the formulation of the first variation of the nondifferentiable convex functions $K(\underset{+}{u_N})$, $W(\epsilon)$, the concept of subdifferentiability must be used. (The method is indicated in [1], p. 31).

Thus among all the kinematically admissible fields, the exact solution is the one, that reduces the potential energy of the structure to an absolute minimum.

4. THE VARIATIONAL INEQUALITY OF COMPLEMENTARY VIRTUAL WORK AND THE MINIMUM OF COMPLEMENTARY ENERGY

A field ψ^0 of stresses σ_{ij}^0 is defined to be statically admissible if it satisfies the condition of equilibrium (9), the statical boundary conditions on Γ_F and the statical conditions imposed by the boundary conditions on Γ_S. Thus the unilateral contact boundary conditions (1), (2) and (8) yield on Γ_S the conditions

$$S_N^0 \le 0, \qquad S_{T_i}^0 = C_{T_i}. \tag{35}$$

Conditions (35) cause the field ψ^0 to be a convex set and thus the variations must be compatible with the convexity of the field ψ^0. The strain field ϵ_{ij}^0 is obtained from σ_{ij}^0 by means of (15). The actual stress field at the position of equilibrium is denoted by σ_{ij} and thus $\sigma_{ij}^0 - \sigma_{ij}$ represents the statically admissible variations.

The combination of eqns (9) and (10) with the boundary conditions on Γ_F and on Γ_U and the Green's identity yields

$$\int_{\Omega} \epsilon_{ij}(\sigma_{ij}^0 - \sigma_{ij})\,d\Omega = \int_{\Gamma_U} U_i(S_i^0 - S_i)\,d\Gamma + \int_{\Gamma_S} [u_N(S_N^0 - S_N) + u_{T_i}(S_{T_i}^0 - S_{T_i})]\,d\Gamma$$
$$\forall\, \sigma_{ij}^0 \in \psi^0. \tag{36}$$

Further, by means of relations analogous to (19), the positive and negative parts of S_N are introduced. The nondecreasing function $-l$ is defined for nonpositive values of S_N, by means of the relation

$$u_N - l(-\underline{S_N}) = 0. \tag{37}$$

Function l is the inverse of k. It can be trivially proved that relations (1), (2) yield the inequality

$$u_N(S_N^0 - S_N) - l(-\underline{S_N})(\underline{S_N^0} - \underline{S_N}) \ge 0 \quad \text{on}\,\Gamma_S\,\forall\,\sigma_{ij}^0 \in \psi^0. \tag{38}$$

Thus combining (36) with inequality (38) the variational inequality

$$\int_{\Omega} \epsilon_{ij}(\sigma_{ij}^0 - \sigma_{ij})\,d\Omega - \int_{\Gamma_S} l(-\underline{S_N})\cdot(\underline{S_N^0} - \underline{S_N})\,d\Gamma - \int_{\Gamma_U} U_i(S_i^0 - S_i)\,d\Gamma \ge 0$$
$$\forall\, \sigma_{ij}^0 \in \psi^0 \tag{39}$$

is obtained. Conservely it can be proved, by following the same procedure with respect to the stress variations, as it has been done in variational inequality (25) with respect to the displacement variations, that relation (39) is equivalent to the strain-displacements relationship (10) and to the boundary conditions on Γ_U and on Γ_S. The variational inequality (39) expresses the "principle" of complementary virtual work when the unilateral contact boundary conditions are holding on Γ_S. The variational inequality is constrained by conditions (35), which cause the variations to be unilateral and make the "principle" of complementary virtual work to hold in the inequality form [4].

For the proof of the minimum theorem of the complementary energy to convex function

$$L(-\xi) = -\int_0^{\xi} l(-\xi)\,\mathrm{d}\xi \tag{40}$$

satisfying the inequality [1].

$$L(-\underline{S}_N^0) - L(-\underline{S}_N) \geq -(\underline{S}_N^0 - \underline{S}_N)\cdot l(-\underline{S}_N) \quad \forall \sigma_{ij}^0 \in \psi^0 \tag{41}$$

is introduced. Moreover from (15) and (17) the inequality

$$W^c(\sigma^0) - W^c(\sigma) \geq \int_\Omega \epsilon_{ij}(\sigma_{ij}^0 - \sigma_{ij})\,\mathrm{d}\Omega \tag{42}$$

arises. The combination of inequalities (41) and (42) with (39) yields the inequality

$$W^c(\sigma^0) + \int_{\Gamma_S} L(-S_N^0)\,\mathrm{d}\Gamma - \int_{\Gamma_U} U_i S_i^0\,\mathrm{d}\Gamma \geq W^c(\sigma) + \int_{\Gamma_S} L(-\underline{S}_N)\,\mathrm{d}\Gamma - \int_{\Gamma_U} U_i S_i\,\mathrm{d}\Gamma \quad \forall \sigma_{ij}^0 \in \psi^0 \tag{43}$$

expressing the theorem of the minimum complementary energy

$$\Pi_c = W^c(\sigma) + \int_{\Gamma_S} L(-S_N)\,\mathrm{d}\Gamma - \int_{\Gamma_U} U_i S_i\,\mathrm{d}\Gamma. \tag{44}$$

The minimum must be sought over the set ψ^0 of admissible stress fields and thus the resulting minimization problem is constrained by eqn (9) and by the inequalities (35). Conversely the variational inequality (39) results from the nonnegativity of the first variations of the complementary energy.

If conditions (6), (7) hold on Γ_S, functions $k(u_N)$ and $l(-\underline{S}_N)$ are zero. For this case as well inequality (25) (resp. (38)) yields by means of the subsidiary condition $u_N^* \leq 0$ (resp. $S_N^0 \leq 0$) the boundary condition (6) and (7).

5. UPPER AND LOWER BOUNDS

By means of relations (1), (2), (8)–(10) and (14) and of Green's identity, the relation

$$\Pi + \Pi_c = \int_\Omega \sigma_{ij}\epsilon_{ij}\,\mathrm{d}\Omega - \int_\Omega f_i u_i\,\mathrm{d}\Omega - \int_{\Gamma_S} (S_N u_N + S_{T_i} u_{T_i})\,\mathrm{d}\Gamma + \int_{\Gamma_F} F_i u_i\,\mathrm{d}\Gamma + \int_{\Gamma_U} \sigma_{ij} n_j u_i\,\mathrm{d}\Gamma = 0 \tag{45}$$

is easily proved. Accordingly the inequality

$$\Pi^* \geq \Pi = -\Pi_c \geq -\Pi_c^0 \tag{46}$$

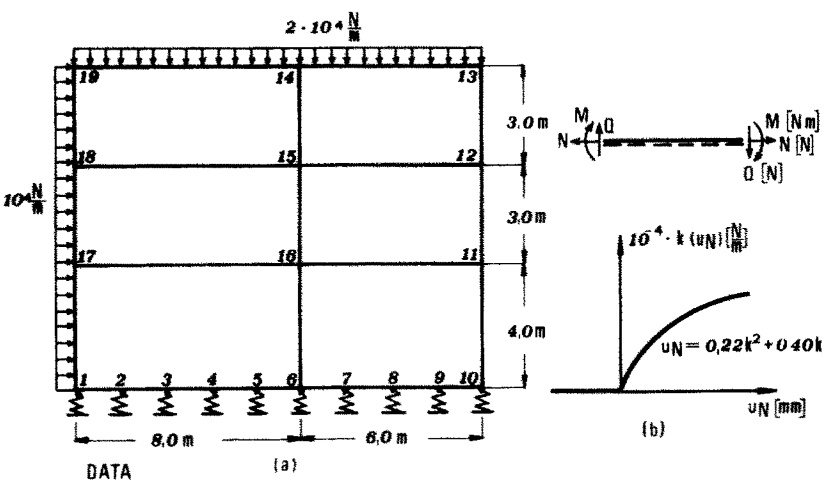
is valid, where Π^* (resp. Π_c^0) represents the potential energy (resp. complementary energy), corresponding to a kinematically (resp. statically) admissible field. Thus Π^* and Π_c^0 provide upper and lower bounds for the energy $\Pi = -\Pi_c$ of the structure at the position of equilibrium and can be used in the approximate analysis of the structure.

6. NUMERICAL EXAMPLE

The theorems of minimum potential and complementary energy make possible the numerical calculation of the exact solution of the unilateral contact boundary value problem. The finite element method is used to discretize the continuum and to establish the kinematically and statically admissible fields in terms of a finite number of unknowns. The potential energy (resp. complementary energy) is expressed as a nonlinear function of the displacements and will be minimized over the kinematically (resp. statically) admissible field. It contains the positive (resp. negative) part of u_N (resp. S_N) and the generally nondifferentiable function $W(\epsilon)$ (resp. $W^c(\sigma)$), a fact, which makes preferable the use of a derivative-free nonlinear programming algorithm. For the solution of the subsequent examples the "flexible tolerence" algorithm of Paviani and Himmelblau [3] has been used. This algorithm is one of the most general nonlinear programming algorithms and gives the minimum of a nonlinear function subjected to nonlinear equality and inequality constraints. The value of the objective function is improved by using "feasible" and "near-feasible points" and thus the computation time is not spent by rigorous feasibility requirements. A positive decreasing function whose form can be found in [3] acts as the tolerence criterion for constraint violation through the entire search and also serves as a criterion for termination of the search. The constraint violation is decreased as the search moves toward the solution. For the unconstrained searches included in the "flexible tolerence" algorithm the simplex method search of Nelder and Mead [8] is used.

In the minimization of the potential (resp. complementary) energy of the discretized continuum the relations between u_N, u_{T_i} and u_i (resp. S_N, S_{T_i} and S_i) at the nodes of the finite elements of the boundary Γ_S, and the subsidiary conditions introduced by the set X^* (resp. ψ^0), define the constraints of the problem. The number of variables is typically large but the numbers of inequalities is small. The speed of convergence depends strongly upon the guessed starting point of the algorithm. This initial solution is obtained, by assuming that the body does not work itself loose from the boundary. This problem is not unilateral and accordingly the solution of a system of equations yields the displacements of stresses which will be used to start the flexible tolerance algorithm. Both following problems are programmed on a UNIVAC 1106 computer. The "flexible tolerance" subroutine given in [3] has been used.

As an example the frame shown in Fig. 2(a) is analyzed. Its modulus of elasticity E is equal to

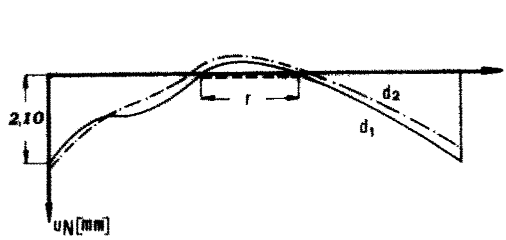


DATA

Horizontal Beams : 40/45 [cm]

Columns : 40/45 [cm]

Foundation Beams : $I = 0{,}01\,m^4, F = 2\,m^2$



(c)

(1)	(2): $10^{-4}\cdot M$	(3): $10^{-4}\cdot M$
1–17	−5,10 −3,20	−7,20 −2,10
17–18	−4,50 −0,20	−4,00 0 0
18–19	8,00 9,50	7,80 9,50
6–16	−7,50 −7,00	−7,10 −6,50
10–11	−3,50 −2,00	−4,50 −1,80

(1): Beam Elements

(2): Classical Method

(3): Unilateral Method

(d)

Fig. 2. A unilateral contact problem for a frame.

$2.05\times 10^{10}\,N/m^2$ if $\epsilon \le 5‰$ and to $1.80\times 10^{10}\,N/m^2$ otherwise. It is assumed that the foundation beam 1–10 of the frame is in unilateral contact with an elastoplastic halfspace, whose properties are concentrated in the $k(u_N)$ function given in Fig. 2(b). The shear stresses S_T on the foundation beam are assumed to be uniformly distributed and equilibrated by the lateral loads of the frame. As a result, the displacement distribution d_1 and the length r of the region, where the foundation beam works itself loose from the halfspace is obtained and is given in Fig. 2(c). The influence of the unilateral contact theory on the bending moments is depicted in Fig. 2(d), where they are compared with the bending moments obtained by the initial guess of the flexible tolerance algorithm. This guess corresponds to the classical method for calculation of foundation structures. The assumption that S_T is uniformly distributed over the whole length of 1–10 does not influence considerably the results of this example.

This can be checked by an iterative corrective procedure. A new uniform distribution of S_T, only over the contact region is assumed, the structure is solved again, and so on until the difference $S_{N_{i+1}} - S_{N_i}$ is made sufficiently small. The corrected displacement-distribution d_2 is given in Fig. 2(c).

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